Mass, momentum and energy flux in water waves

By G. B. WHITHAM

Mathematics Department, Massachusetts Institute of Technology

(Received 20 March 1961)

This paper gives a direct derivation of some results obtained by Longuet-Higgins & Stewart (1960, 1961) on the amplitude variation of waves propagating on a non-uniform stream. The derivation raises a number of subtle points in the use of the equations of conservation of mass, momentum and energy. These points are of general interest and are discussed in detail with other applications, since they do not seem to have been pointed out previously.

1. Introduction

Longuet-Higgins & Stewart (1960, 1961) use a perturbation analysis to find solutions for waves propagating on a given non-uniform stream. The results for the amplitude variations in terms of the changes in stream velocity are found to satisfy a simple energy-balance equation. However, only certain of the conceivable terms appear in this relation, and it is noted that earlier writers have often included the wrong ones in trying to write it down directly. The authors give further discussion of the difficulty of deciding between the various terms and conclude that no unique answer is given by physical intuition.

Now, it is true that further information is needed, but the corresponding conservation equations for momentum and mass suffice. The required result can be obtained from the full set of conservation equations in a straightforward way and without appeal to any deep intuition. However, some change of interpretation and viewpoint is involved.

When the conservation equations were studied in this connexion, a number of points which are easily overlooked were noticed. (In fact several mistakes were made at first!) It seemed, therefore, that a careful derivation and examination of the conservation equations would be of general use, since the standard references on water waves do not seem to have a full account. This is given first in §§ 2 and 3, with illustrative examples in §§ 4 and 5, before going on to the results for non-uniform currents and streams in § 6.

Longuet-Higgins & Stewart (1960) show that the energy flux for waves of amplitude a moving along a main stream whose 'mass transport velocity' is U_m (the distinction between the various definitions of main-stream velocities is noted later) is given by

$$\tfrac{1}{2}\rho h U_m^3 + U_m S + (U_m + c_g) E, \tag{1}$$

where

$$E = \tfrac{1}{2}\rho g a^2, \tag{2}$$

$$S = \left(\frac{2c_g}{c} - \frac{1}{2}\right) E. \tag{3}$$

Here, c and c_g are the phase and group velocities for waves moving into fluid at rest, h is the mean depth, ρ is the density and g is the acceleration of gravity. For a non-uniform stream $U_m(x)$, it is found that the amplitude variation is given correctly by the relation

$$\frac{d}{dx}\{(U_m+c_g)E\}+S\frac{dU_m}{dx}=0. \tag{4}$$

The above authors interpret the first term in (1) as main stream energy and the third term as the usual propagation of wave energy, density E, with group velocity U_m+c_g. The quantity S is called the radiation stress and the term U_mS in (1) is interpreted as the work done by the current U_m against the radiation stress of the waves. Discussion then centres around why only part of the change $d(U_mS)/dx$ appears in (4).

It seems to the present writer that the difficulties are due to trying to divide (1) into 'main stream energy' and 'wave energy'. In this problem U_m depends on the amplitude of the waves so that the first term of (1) can not be dismissed simply as main-stream energy. Also U_mS can not really be counted as a contribution to wave energy because the group velocity (wave-energy flux divided by wave-energy density) must certainly be U_m+c_g not $U_m(S/E)+U_m+c_g$. It is true that in simpler cases one can usually make the division into main stream and wave energy in a straightforward way. But this division is just equivalent to subtracting a multiple of the momentum equation from the full energy equation to leave only 'wave energy'. In a complicated case one can write down the full energy and momentum equations and take suitable combinations. With this point of view (4) can easily be derived. In fact, it turns out that it is the main stream terms that require special care, not the U_mS term. S is just the principal term in the momentum flux.

Perhaps these arguments are seen most convincingly by looking at a roughly analogous situation for the dynamics of a system of particles. For particles of mass m and velocity $\mathbf{v}$, with forces $\mathbf{F}$, we have the momentum and energy relations

$$\Sigma m\dot{\mathbf{v}}=\Sigma\mathbf{F},\quad \frac{d}{dt}\Sigma\tfrac{1}{2}m\mathbf{v}^2=\Sigma\mathbf{F}.\mathbf{v}.$$

If $\mathbf{v}=\mathbf{U}+\mathbf{u}$, the energy relation becomes

$$\frac{d}{dt}\{\tfrac{1}{2}\mathbf{U}^2\Sigma m+\mathbf{U}.(\Sigma m\mathbf{u})+\Sigma\tfrac{1}{2}m\mathbf{u}^2\}=\Sigma\mathbf{F}.(\mathbf{U}+\mathbf{u}), \tag{5}$$

and the analogy between the left-hand side and (1) is immediate. But, if $\mathbf{U}$ is constant, a suitable multiple of the momentum equation is subtracted to give

$$\frac{d}{dt}\{\Sigma\tfrac{1}{2}m\mathbf{u}^2\}=\Sigma\mathbf{F}.\mathbf{u}, \tag{6}$$

which is the energy equation for an observer moving with velocity $\mathbf{U}$. Suppose next that $\mathbf{U}$ is a function of time. Then, taking the scalar product of $\mathbf{U}$ with the momentum equation, we have

$$\mathbf{U}.\Sigma m(\dot{\mathbf{U}}+\dot{\mathbf{u}})=\Sigma\mathbf{F}.\mathbf{U}. \tag{7}$$

Subtracting this from (5), we obtain

$$\frac{d}{dt}\{\tfrac{1}{2}\Sigma m\mathbf{u}^2\} + (\Sigma m\mathbf{u}) . \frac{d\mathbf{U}}{dt} = \Sigma\mathbf{F} . \mathbf{u}. \tag{8}$$

The left-hand side is directly analogous to (4). This shows that correctly viewed the water wave results are not surprising. For a moving system, a multiple of the momentum always appears in the energy equation (as in (1) and (5)), and for an accelerated motion the typical momentum multiplied by the acceleration appears (as in (4) and (6)). It is true that the extra term in (8) can be called part of the rate of working by the 'radiation stress' $\Sigma m\mathbf{u}$, but it is more familiar as the rate of working by the 'fictitious forces', $-m\,d\mathbf{U}/dt$, of the accelerated system.

2. Stokes's waves to second order for water of finite depth

The quantities appearing in the equations of conservation of mass, momentum and energy involve terms of the second order in the amplitude. For many purposes the required manipulation of these equations can be done before evaluating the various integrals in terms of a, and in the final form the linear theory suffices even though the quantities concerned are $O(a^2)$. This is true, for example, for the energy equation for waves moving into water at rest, but not for the momentum equation. In all cases, it is reassuring to have expressions for all the flow quantities to order a^2. These are provided by Stokes's expansions in powers of a for periodic waves.

For two-dimensional flow, the velocity potential $\Phi(x, y, t)$ satisfies Laplace's equation and the pressure p is given by Bernoulli's equation

$$(p/\rho) + \Phi_t + \tfrac{1}{2}(\nabla\Phi)^2 + gy = B(t), \tag{9}$$

where y is measured vertically upwards and $B(t)$ is an arbitrary function of t. One usually says that $B(t)$ can be absorbed into Φ and then forgets about it. Here it is important. In this problem a certain term A, proportional to a^2 (see (15) below), has to appear in one of three places: (i) on the right of (9), (ii) in Φ as a term proportional to t, or (iii) as the difference between $y = 0$ and the mean position of the surface. On the whole it seems to be most convenient to have it in Φ, even at the expense of losing the periodicity of Φ. In fact a casual assumption of the periodicity of Φ is the easiest way of getting (1) wrong. Accordingly then, $B(t)$ is set equal to zero, the average elevation is taken to be $y = 0$ and a term proportional to t is included in Φ. Without any loss, we can take $p = 0$ at the surface.

The boundary conditions on Φ are

$$\left.\begin{aligned} \Phi_t + \tfrac{1}{2}(\nabla\Phi)^2 + gy &= 0, \\ \eta_t + \Phi_x\eta_x - \Phi_y &= 0, \end{aligned}\right\} \quad \text{at} \quad y = \eta(x, t), \tag{10}$$

and

$$\Phi_y = 0 \quad \text{at} \quad y = -h; \tag{11}$$

$y = \eta(x,t)$ is the elevation of the surface and the depth h is taken to be constant Then, the general periodic solution correct to terms in a^2 is

$$\eta = a\cos(kx-\omega t) + a^2\mu_1 \cos 2(kx-\omega t), \tag{12}$$

$$\left.\begin{aligned} \Phi &= Ux - (\tfrac{1}{2}U^2 + A)\,t + \phi, \\ \phi &= \frac{a\omega_0}{k}\frac{\cosh k(y+h)}{\sinh kh}\sin(kx-\omega t) + a^2\mu_2 \cosh 2k(y+h)\sin 2(kx-\omega t), \end{aligned}\right\} \tag{13}$$

where
$$\left.\begin{aligned} \omega &= Uk + \omega_0, \quad \omega_0^2 = gk\tanh kh, \\ c^2 &= \frac{\omega_0^2}{k^2} = \frac{g}{k}\tanh kh, \quad c_g = \frac{d\omega_0}{dk} = \tfrac{1}{2}c\left(1 + \frac{2kh}{\sinh 2kh}\right), \end{aligned}\right\} \tag{14}$$

and
$$A = \frac{1}{2}\frac{gk}{\sinh 2kh}a^2 = \frac{E}{\rho h}\left(\frac{c_g}{c} - \frac{1}{2}\right), \tag{15}$$

$$\left.\begin{aligned} \mu_1 &= \tfrac{1}{2}k\coth kh\left(1 + \frac{3}{2\sinh^2 kh}\right), \\ \mu_2 &= \frac{3}{8}\frac{\omega_0}{\sinh^4 kh}. \end{aligned}\right\} \tag{16}$$

The quantity A is the only second-order term that will be needed. Notice that it vanishes in the case of deep-water waves since $c_g = \frac{1}{2}c$.

The pressure is given by

$$\frac{p}{\rho} = A - \phi_t - U\phi_x - \tfrac{1}{2}(\nabla\phi)^2 - gy. \tag{17}$$

Hence, the mean value (averaged over a period) is

$$\begin{aligned} \frac{\overline{p}}{\rho} &= A - \tfrac{1}{2}\overline{(\nabla\phi)^2} - gy, \\ &= -gy - a^2\frac{gk}{\sinh 2kh}\sinh^2 k(y+h). \end{aligned} \tag{18}$$

The velocity U in the solution (13) is the mean value of the x-velocity, i.e.

$$U = \overline{\Phi_x},$$

and it is very important to distinguish this from the mass transport velocity U_m defined such that the mean mass flux across a vertical section is $\rho h U_m$. It should also be noted that U and h may themselves depend on a and k in certain problems; they may differ from the undisturbed values by terms of order a^2. Examples are given in §§ 4 and 5. The only term affected is the $Ux - \frac{1}{2}U^2 t$ term in Φ; elsewhere in the formulae (12)–(16), the undisturbed values can be used since the error is $O(a^3)$. Finally, the usual restrictions on the parameter ranges for Stokes's waves to apply should be borne in mind. In particular, for long waves the condition is that a/h should be less than a certain multiple of $(kh)^2$.

3. Mass, momentum, energy

The equations for the conservation of mass, momentum and energy can be written down directly or established from $\nabla^2\Phi = 0$ and the boundary conditions (10) and (11). They take the form

$$\frac{\partial P}{\partial t} + \frac{\partial Q}{\partial x} = 0, \tag{19}$$

where P, Q are given by the following integrals.

(i) Mass:
$$P_0 = \int_{-h}^{\eta} \rho\, dy, \quad Q_0 = \int_{-h}^{\eta} \rho \Phi_x dy. \tag{20}$$

(ii) Momentum:
$$P_1 = \int_{-h}^{\eta} \rho \Phi_x dy, \quad Q_1 = \int_{-h}^{\eta} (p + \rho \Phi_x^2)\, dy. \tag{21}$$

(iii) Energy:
$$P_2 = \int_{-h}^{\eta} \{\tfrac{1}{2}\rho(\nabla\Phi)^2 + \rho g(y+H)\}\, dy, \tag{22}$$

$$Q_2 = \int_{-h}^{\eta} \{p + \tfrac{1}{2}\rho(\nabla\Phi)^2 + \rho g(y+H)\}\, \Phi_x dy. \tag{23}$$

The potential energy is measured from $y = -H$. We have already defined $y = 0$ to be the mean level of the surface and this may be at different heights in different regions (for example, on the two sides of an obstacle in the surface). Therefore, an arbitrary level $y = -H$ is chosen for the potential energy rather than $y = 0$.

We now calculate the mean values of the above integrals from the formulae of § 2. It should be noted that $\overline{\eta} = 0$, and ϕ and all its derivatives have zero mean values. Evaluations like the following are used repeatedly:

$$\overline{\int_{-h}^{\eta} \phi_x dy} = \int_{-h}^{0} \overline{\phi_x} dy + \overline{\eta[\phi_x]_{y=0}} + O(a^3)$$
$$= \overline{\eta[\phi_x]_{y=0}} + O(a^3), \tag{24}$$

$$\overline{\int_{-h}^{\eta} \phi_x^2 dy} = \int_{-h}^{0} \overline{\phi_x^2} dy + O(a^3). \tag{25}$$

Mass:

We have
$$\overline{P}_0 = \overline{\rho(h+\eta)} = \rho h. \tag{26}$$

$$\overline{Q}_0 = \overline{\rho U(h+\eta)} + \rho \overline{\int_{-h}^{\eta} \phi_x dy},$$
$$= \rho U h + \rho \overline{\eta[\phi_x]_{y=0}}$$
$$= \rho U h + E/c, \tag{27}$$

from (12), (13), (14); as before $E = \frac{1}{2}\rho g a^2$. Although Φ_x oscillates about U, there is greater depth at the higher velocities and consequently the mass flux is greater than $\rho U h$. The 'mass transport velocity' U_m is defined by

$$U_m = \frac{\overline{Q}_0}{\overline{P}_0} = U + \frac{E}{\rho h c}. \tag{28}$$

Momentum:

We have
$$\overline{P}_1 = \overline{Q}_0 = \rho U h + E/c = \rho h U_m, \tag{29}$$

and, from (17),

$$Q_1/\rho = \int_{-h}^{\eta} \{U^2 + A - \phi_t + U\phi_x + \tfrac{1}{2}\phi_x^2 - \tfrac{1}{2}\phi_y^2 - gy\}\, dy,$$

so
$$\overline{Q}_1/\rho = hU^2 + hA - \overline{\eta[\phi_t]_{y=0}} + \overline{U\eta[\phi_x]_{y=0}} + \frac{1}{2}\int_{-h}^{0} \overline{(\phi_x^2 - \phi_y^2)}\, dy - \tfrac{1}{2}g\overline{\eta^2} + \tfrac{1}{2}gh^2.$$

Using the formulae (12)–(16), we obtain

$$\bar{Q}_1 = \rho h U^2 + \tfrac{1}{2}\rho g h^2 + \frac{E}{c}(2U + 2c_g - \tfrac{1}{2}c)$$
$$= \rho h U_m^2 + \tfrac{1}{2}\rho g h^2 + E\left(\frac{2c_g}{c} - \frac{1}{2}\right), \quad (30)$$

to second order.

Energy:

We have

$$\bar{P}_2/\rho = \overline{\int_{-h}^{\eta} \{\tfrac{1}{2}U^2 + U\phi_x + \tfrac{1}{2}(\nabla\phi)^2 + g(y+H)\}\,dy}$$
$$= \tfrac{1}{2}hU^2 + U\overline{\eta[\phi_x]_{y=0}} + \int_{-h}^{0} \tfrac{1}{2}\overline{(\nabla\phi)^2}\,dy + \tfrac{1}{2}g\overline{\eta^2} - \tfrac{1}{2}gh^2 + ghH,$$
$$\bar{P}_2 = \tfrac{1}{2}\rho h U^2 + \rho g h(H - \tfrac{1}{2}h) + E(1 + U/c)$$
$$= \tfrac{1}{2}\rho h U_m^2 + \rho g h(H - \tfrac{1}{2}h) + E. \quad (31)$$

When the expression (17) for p is substituted in (23), we have simply

$$Q_2/\rho = \int_{-h}^{\eta} \{\tfrac{1}{2}U^2 + gH + A - \phi_t\}(U + \phi_x)\,dy, \quad (32)$$

and
$$\bar{Q}_2 = (\tfrac{1}{2}U^2 + gH + A)\bar{Q}_0 - \rho U\overline{\eta[\phi_t]_{y=0}} - \int_{-h}^{0} \rho\overline{\phi_t\phi_x}\,dy$$
$$= \rho h U(\tfrac{1}{2}U^2 + gH) + (E/c)\{\tfrac{3}{2}U^2 + gH + U(2c_g - \tfrac{1}{2}c) + c(U + c_g)\}$$
$$= \rho h U_m(\tfrac{1}{2}U_m^2 + gH) + U_m E\{(2c_g/c) - \tfrac{1}{2}\} + E(U_m + c_g). \quad (33)$$

It should be noted that in all the above calculations, A is the only second-order quantity needed; otherwise it is sufficient to know $\bar{\phi}_t = \bar{\phi}_x = 0$, etc.

The various expressions are collected in the following table:

	Density P	Flux Q
(0) Mass	ρh	$\rho h U_m = \rho h(U + E/\rho c h)$
(1) Momentum	$\rho h U_m$	$\rho h U_m^2 + \tfrac{1}{2}\rho g h^2 + E\{(2c_g/c) - \tfrac{1}{2}\}$
(2) Energy	$\rho h\{\tfrac{1}{2}U_m^2 + g(H - \tfrac{1}{2}h)\} + E$	$\rho h U_m\{\tfrac{1}{2}U_m^2 + gH\} + U_m E\{(2c_g/c) - \tfrac{1}{2}\} + E(U_m + c_g)$

The above results can be combined to give

$$\bar{P}_3 \equiv \bar{P}_2 - U_m\bar{P}_1 + \{\tfrac{1}{2}U_m^2 - g(H - \tfrac{1}{2}h)\}\bar{P}_0 = E, \quad (34)$$
$$\bar{Q}_3 \equiv \bar{Q}_2 - U_m\bar{Q}_1 + \{\tfrac{1}{2}U_m - g(H - \tfrac{1}{2}h)\}\bar{Q}_0 = E(U_m + c_g). \quad (35)$$

These quantities could be called the density and flux of 'excess energy' or 'wave energy'. For the uniform flow considered so far, all the quantities are constants and, for example, constant energy flux can be expressed equally well as total energy $\bar{Q}_2$ = constant or wave energy $\bar{Q}_3$ = constant. However, applications are to problems where (i) the flow conditions and wave properties vary slowly over distances and times large compared with the wavelength and period, or (ii) a number of different regions of uniform flow occur. In the former case, (19) is used for the mean quantities, i.e.

$$\frac{\partial\bar{P}_i}{\partial t} + \frac{\partial\bar{Q}_i}{\partial x} = 0 \quad (i = 0, 1, 2). \quad (36)$$

But, it is obvious that the 'wave energy' quantities $\bar{P}_3$, $\bar{Q}_3$ do not satisfy this conservation equation in general, because the coefficients U_m, $\{\frac{1}{2}U_m^2 - g(H-\frac{1}{2}h)\}$ will not be constant. Hence, additional terms such as the last one in (4) are to be expected. A similar comment applies to problems in which different uniform regions are connected in some way, as in propagation over a step or past an obstacle. In such problems the discontinuity conditions or 'shock conditions' corresponding to (36) are used. In all cases, the recommended procedure is to use the equations (36) for the total quantities and then take suitable combinations to simplify.

It should be noted that the mean quantities in (36) involve four basic variables U_m, h, k, E. To complete the system of equations we add the kinematic equation

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0. \tag{37}$$

In the applications to non-uniform flows, k and ω vary with x and t, and (39) expresses the conservation of waves. (Alternatively if the phase is $\theta(x,t)$, $\omega = -\partial\theta/\partial t$, $k = \partial\theta/\partial x$; therefore (37) follows.) The four equations in (36) and (37), or equivalent ones, are the tools used in solving the problems in the next sections. It is interesting to note that these form a hyperbolic system for the propagation of important physical quantities as opposed to Laplace's equation for the full details of the flow. We return to this question in § 7.

One of the main points of this paper is that U_m (or equivalently U) and h have to be treated as unknowns, as the next sections show.

4. Propagation over a smooth step

In this section and the next, two standard examples are noted to illustrate points in using the conservation equations. First, consider an initially uniform wave train passing over a 'smooth' step, i.e. a gradual change in depth between two uniform regions. The problem is to determine the transmitted wave train in terms of the incident one. If the change in depth is sufficiently gradual (i.e. changes in depth take place over length scales much greater than typical wavelengths), it may be assumed that the reflected energy is negligible.

The purpose of this example is to show that the mean velocity U has to be considered with care. Suppose we consider the case in which the fluid was initially at rest before the arrival of the wave train. Then, arguing naïvely, we might perhaps say: (i) U can be set equal to zero throughout, (ii) the frequency is constant so that the change in wave number is given by

$$(gk \tanh kh)_1 = (gk \tanh kh)_2, \tag{38}$$

where subscripts 1 and 2 refer to the incident and transmitted waves, (iii) the energy flux is the same on the two sides so that the amplitude change is given by

$$(Ec_g)_1 = (Ec_g)_2. \tag{39}$$

The answers (38) and (39) are in fact correct, but some questions arise if we look a little further. If $U = 0$, there is a mass transport E/c in each wave; hence, presumably

$$(E/c)_1 = (E/c)_2, \tag{40}$$

to conserve mass. But this contradicts (39), and we have the problem of finding which one is correct.

The error is in the casual assumption that $U = 0$ everywhere. We are certainly at liberty to prescribe that $U = 0$ in the incident wave, but the transmitted wave has to be determined and we must allow $U_2 \neq 0$. Then (40) becomes

$$(E/c)_1 = (E/c)_2 + \rho h_2 U_2, \tag{41}$$

and actually determines the unknown mean velocity U_2 required to balance the incident mass transport. U_2 is second order in the amplitude and does not modify (38) or (39) to the order of approximation being used. The additional terms in the energy flux are all $O(E^2)$, since $U_m = O(E)$ and any change in the height h of the mean surface above the horizontal level will be $O(E)$.

The moral is that the mean velocity U can not be prescribed in advance as one might expect. The same applies to the mean depth h, as will be seen in the next example; variations $O(a^2)$ must be allowed. Of course in (38) and in all the formulae in (14), this change in h is negligible and h can be taken there as the undisturbed depth.

The kinematic condition (38), corresponding to (37), and two of the conservation equations (36) were used above. The third conservation equation, the momentum flux, gives the force on the step.

5. Obstacle in a steady stream

Another similar example concerns the wave train produced by a two-dimensional obstacle in the surface of a steady stream. Only gravity waves are considered and they are formed downstream of the obstacle. The mean undisturbed velocity U_0 and depth h_0 upstream of the obstacle can be prescribed, but the downstream values U and h can not. They differ from the upstream values by terms proportional to E. The flow pattern is assumed to be steady;† hence, the frequency $\omega = \omega_0 - Uk = 0$ and we have $c(k) = U$. Consistent with the approximations adopted, U can be replaced by U_0 in *this* relation. Thus k is determined by

$$c(k) = U_0. \tag{42}$$

Again, this corresponds to the kinematic relation (37).

The relations for the flux of mass, momentum and energy give:

Mass:
$$\rho h U_m = \rho h_0 U_0, \tag{43}$$

Momentum:
$$\rho h U_m^2 + \tfrac{1}{2}\rho g h^2 + E\{(2c_g/c) - \tfrac{1}{2}\} + R = \rho_0 h_0 U_0^2 + \tfrac{1}{2}\rho g h_0^2, \tag{44}$$

Energy:
$$\tfrac{1}{2}\rho h U_m^3 + \rho g h^2 U_m + U_m E\{(2c_g/c) - \tfrac{1}{2})\} + E(U_m - c_g) = \tfrac{1}{2}\rho h_0 U_0^3 + \rho g h_0^2 U_0. \tag{45}$$

Here R is the 'wave resistance' experienced by the obstacle. Also H is taken equal to h in the energy flux formula, since the bottom is horizontal.

† In the formulae of the previous sections, U and U_m were reckoned positive in the direction of wave propagation. For this application, the directions are opposite so $-U$ and $-U_m$ are substituted to get the formulae given below.

Solving these relations correct to order E, we find

$$RU_0 = E(U_0 - c_g). \tag{46}$$

This is the usual formula for wave resistance and is more easily deduced directly (see Lamb 1932, Art. 249) by applying the energy argument in the frame of reference in which the obstacle moves with velocity U_0 into fluid at rest. The left-hand side is the work done in moving the obstacle and the right-hand side is the rate of gain of energy in the wave train. In that derivation, the change in mean velocity and mean depth need not be considered. It is interesting to note that the mass and momentum equations do not give further information about the wave resistance, but serve to determine the small changes in U and h. This example also serves as a warning; all kinds of spurious results can be deduced if U and h are mistakenly taken equal to U_0 and h_0. In the present case, (43) and (45) would give $E = 0$, i.e. no waves, and (44) would give an erroneous answer for R.

6. Waves on non-uniform streams and currents

Here the problems discussed by Longuet-Higgins & Stewart are reconsidered and their results derived directly from the conservation laws. We consider waves propagating along a prescribed steady current. The current is specified by giving the velocity and depth. We must distinguish between the undisturbed velocity $U_0(x)$ before the waves are present, the mean velocity $U(x)$ with the wave train and the mass flow velocity $U_m(x)$. Presumably any one of these could be measured and prescribed. They differ from each other by terms $O(a^2)$, and, in view of the warnings of the last two sections, we must allow U to differ from U_0. Likewise the undisturbed depth $h_0(x)$ without the waves must be distinguished from the mean depth $h(x)$ with the waves. The depth and velocity are not independent, since the pressure must vanish at the surface; for example

$$\tfrac{1}{2}U_0^2(x) + gh_0(x) = \text{constant}.$$

The change in velocity with x must be adjusted by inflow and we consider: (i) inflow from below, (ii) inflow from the sides.

(i) *Current fed from below*

The fluid entering the current across the lower boundary $y = -h$ is assumed to have a horizontal velocity U^*. Probably the appropriate assumption here is that U^* is equal to the mean velocity U. However, the same result is obtained for any U^* which differs from U by terms of order E. So we can include, for example, the possibilities $U^* = U_0$ or $U^* = U_m$ at the same time.

The change in mass flux between the sections x and $x+\delta x$ is $\delta(\rho h U_m)$ and this must be the inflow into the section from below. Thus the vertical inflow velocity is $V = d(hU_m)/dx$. We next obtain the expressions for the change in momentum and energy between the sections x and $x+\delta x$. The momentum added by the inflow is $U^*\delta(\rho h U_m)$; hence the equation for momentum balance becomes

$$\delta\{\rho h U_m^2 + \tfrac{1}{2}\rho g h^2 + S\} = U^*\delta(\rho h U_m), \tag{47}$$

where S is the 'excess momentum flux'

$$S = E\{(2c_g/c) - \tfrac{1}{2}\}. \tag{48}$$

The kinetic energy carried in from the fluid below† is $\frac{1}{2}U^{*2}\delta(\rho h U_m)$; the potential energy is zero because the choice $H = h$ makes the bottom of the current the zero level for potential energy. The work done against the mean pressure $\rho g h$ in introducing the fluid is $\rho g h \delta(U_m h)$. Therefore the equation for energy balance becomes

$$\delta\{\tfrac{1}{2}\rho h U_m^3 + \rho g h^2 U_m + U_m S + E(U_m + c_g)\} = (\tfrac{1}{2}U^{*2} + gh)\,\delta(\rho h U_m). \tag{49}$$

Now we take U_m times (47) away from (49) to get

$$\delta\{E(U_m + c_g)\} + S\delta U_m = \tfrac{1}{2}(U^* - U_m)^2\,\delta(\rho h U_m). \tag{50}$$

Since we are assuming that the various U's differ by terms of order E, the right-hand side can be neglected; also, U_m can be approximated by U in the terms on the left. Then

$$\frac{d}{dx}\{E(U + c_g)\} + S\frac{dU}{dx} = 0. \tag{51}$$

Equally well U could now be replaced by U_0 since the error would again be $O(E^2)$. The relation (51) is the result obtained by Longuet-Higgins & Stewart after detailed analysis of the full solution.

The equation for changes in wave-number comes from $\omega =$ constant, i.e.

$$\frac{d}{dx}\{Uk + (gk\tanh kh)^{\frac{1}{2}}\} = 0; \tag{52}$$

then (51) determines the change in amplitude.

(ii) *Current fed from the sides*

Consider the same problem of a non-uniform stream $U(x)$, but this time with finite breadth b and with inflow on the sides to balance the changes in mean flow. Between two sections x, $x + \delta x$ the mass inflow is $b\delta(\rho h U_m)$. We assume that the inflow is uniform with depth, i.e. the inflow velocity is

$$-W = bh^{-1}d(hU_m)/dx \tag{53}$$

independent of y, and that at any depth the incoming fluid has the momentum and energy of the fluid in the current at that depth. (Other assumptions could be appropriate but this is the case considered by Longuet-Higgins & Stewart.)

The x-momentum added by the inflow is

$$-W\,\delta x\,\overline{\int_{-h}^{\eta} \rho(U + \phi_x)\,dy} = -\rho h U_m W\,\delta x. \tag{54}$$

The energy added is

$$-W\,\delta x\,\overline{\int_{-h}^{\eta} \rho\{g(h+y) + \tfrac{1}{2}(\nabla\Phi)^2\}\,dy},$$

† The contribution $\frac{1}{2}V^2$ to the kinetic energy is second order in dU^*/dx and may be neglected.

and the work done by the pressure is

$$-W\,\delta x\,\overline{\int_{-h}^{\eta} p\,dy}.$$

Thus the total contribution to the energy change is

$$\begin{aligned}-W\,\delta x\,\overline{\int_{-h}^{\eta}\{p+\rho g(h+y)+\tfrac{1}{2}\rho(\nabla\Phi)^2\}\,dy}\\ &= -W\,\delta x\,\overline{\int_{-h}^{\eta}\{\rho gh+A+\tfrac{1}{2}U^2-\rho\phi_t\}\,dy}\\ &= -W\,\delta x\{(\rho gh+A+\tfrac{1}{2}U^2)\,h-\rho\overline{\eta[\phi_t]_{y=0}}\}\\ &= -W\,\delta x\{\tfrac{1}{2}\rho hU_m^2+\rho gh^2+E[(c_g/c)-\tfrac{1}{2}]+E\}.\end{aligned} \quad (55)$$

The momentum equation for the current is

$$b\delta\{\rho hU_m^2+\tfrac{1}{2}\rho gh^2+S\} = -\rho hU_m W\,\delta x, \quad (56)$$

and the energy equation is

$$b\delta\{\tfrac{1}{2}\rho hU_m^3+\rho gh^2U_m+U_mS+E(U_m+c_g)\} = -\{\tfrac{1}{2}\rho hU_m^2+\rho gh^2+T+E\}\,W\,\delta x, \quad (57)$$

where

$$T \equiv E\{(c_g/c)-\tfrac{1}{2}\}. \quad (58)$$

We now take U_m times (56) away from (57) to give

$$\begin{aligned}\delta\{E(U_m+c_g)\}+S\,\delta U_m+(E+T)\,b^{-1}\,W\,\delta x\\ &= -\rho(gh-\tfrac{1}{2}U_m^2)\{hWb^{-1}\delta x+\delta(hU_m)\}.\end{aligned} \quad (59)$$

The right-hand side is zero, by (53); hence we have

$$\frac{d}{dx}\{E(U_m+c_g)\}+E\frac{W}{b}+S\frac{dU_m}{dx}+T\frac{W}{b} = 0. \quad (60)$$

Since U_m and h depend on x only, W would increase linearly with z across the current and W/b can be replaced by dW/dz. Also, E is independent of z, so we can write

$$\frac{d}{dx}\{E(U_m+c_g)\}+\frac{d(EW)}{dz}+S\frac{dU_m}{dx}+T\frac{dW}{dz} = 0. \quad (61)$$

The first two terms are the divergence of the simple energy flux vector with components $E(U_m+c_g)$, EW; the last two terms give the modification due to the non-uniform stream. Equation (61) is the form given by Longuet-Higgins & Stewart.

When (53) is substituted in (60), we have

$$\frac{d}{dx}\{E(U_m+c_g)\}+S\frac{dU_m}{dx} = \frac{(E+T)}{h}\frac{d(U_mh)}{dx}. \quad (62)$$

Since all the terms are now proportional to E, U_m can be replaced by U or U_0 and h by h_0, to the same order of approximation. The variation of k is again given by (52).

In both cases, it seems clear that the results will be modified if different assumptions are made as to the energy and momentum of the incoming fluid. For example, it might be appropriate to assume that the fluid added to the current from the sides has just the mean flow energy $\frac{1}{2}gh+\frac{1}{2}U^2$ per unit mass.

7. Propagation of changes in mass, momentum, energy and wave-number

We now consider the question, raised at the end of § 3, of the wave propagation specified by equations (36) and (37). As the previous examples show, U_m, h, k, E should be considered as the basic variables determined by these equations. The previous sections have concerned time-independent solutions for these quantities. We now consider time-dependent propagation.

The equations are

$$\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}(hU_m) = 0, \tag{63}$$

$$\frac{\partial}{\partial t}(hU_m)+\frac{\partial}{\partial x}(hU_m^2+\tfrac{1}{2}gh^2+S) = 0, \tag{64}$$

$$\frac{\partial}{\partial t}(\tfrac{1}{2}U_m^2h+\tfrac{1}{2}gh^2+E)+\frac{\partial}{\partial x}\{\tfrac{1}{2}U_m^3h+gh^2U_m+U_mS+E(U_m+c_g)\} = 0, \tag{65}$$

$$\frac{\partial k}{\partial t}+\frac{\partial \omega}{\partial x} = 0. \tag{66}$$

The simplest case is to assume that the water is initially undisturbed and of uniform depth h_0. This means that U_m and $h-h_0$ are due entirely to the wave motion and are both $O(E)$. Then, linearizing the above equations to first order in E and in changes in k, we have

$$\left.\begin{aligned}\frac{\partial h}{\partial t}+h_0\frac{\partial U_m}{\partial x} &= 0,\\ h_0\frac{\partial U_m}{\partial t}+gh_0\frac{\partial h}{\partial x}+\frac{\partial S}{\partial x} &= 0,\\ \frac{\partial E}{\partial t}+c_g\frac{\partial E}{\partial x} &= 0,\\ \frac{\partial k}{\partial t}+c_g\frac{\partial k}{\partial x} &= 0.\end{aligned}\right\} \tag{67}$$

The last equation is completely uncoupled in this linearized set, and shows that changes in k propagate with the group velocity c_g. The third equation shows that energy changes also propagate with the group velocity, as we would expect. The momentum S is simply proportional to E so we have

$$S = f_1(x-c_gt). \tag{68}$$

Then the general solution of the first two equations in (67) is

$$h-h_0 = -\frac{1}{gh_0-c_g^2}f_1(x-c_gt)+f_2(x-[gh_0]^{\frac{1}{2}}t)+f_3(x+[gh_0]^{\frac{1}{2}}t), \tag{69}$$

$$U_m = -\frac{c_g}{gh_0-c_g^2}f_1(x-c_gt)+(g/h_0)^{\frac{1}{2}}f_2(x-[gh_0]^{\frac{1}{2}}t)-(g/h_0)^{\frac{1}{2}}f_3(x+[gh_0]^{\frac{1}{2}}t). \tag{70}$$

The additional wave speeds $\pm(gh_0)^{\frac{1}{2}}$ are the speeds for shallow-water waves, and the f_2 and f_3 terms are exactly the same as in shallow-water theory. But the additional terms f_1, proportional to the energy, appear due to the coupling through the $\partial S/\partial x$ term in the second equation of (67). More generally we may note that (63) and (64) are exactly the non-linear shallow-water equations with the additional momentum term $\partial S/\partial x$. Since the length scale for changes in U_m and h was assumed to be large compared with the wavelength $2\pi/k$, it is not surprising that direct changes in these quantities propagate with the *long-wave* speeds $\pm (gh_0)^{\frac{1}{2}}$. However, the more interesting results are the changes in h and U_m accompanying the amplitude changes.

This type of propagation seems to be worth exploring further but needs a fuller investigation of the quantities $\bar{P}_i$, $\bar{Q}_i$, k, ω and their dependence on U_m, h, k, a. For example, it would be inconsistent to retain the non-linear terms in (63)–(66) for this problem, since they are $O(E^2)$ and the densities and fluxes were not calculated to this order. Again, if terms in a^3 are included in the Stokes waves, ω is found to depend on a^2. This must also be included in a more accurate discussion; it may introduce interesting effects in coupling the equation for wave-number with the other equations. Such investigations go beyond the scope of the present paper.

This research was supported by the U.S. Air Force under Contract AF 49(638)–708, monitored by the Air Force Office of Scientific Research of the Air Research and Development Command.

REFERENCES

LAMB, H. 1932 *Hydrodynamics*, 6th ed. Cambridge University Press.
LONGUET-HIGGINS, M. S. & STEWART, R. W. 1960 *J. Fluid Mech.* **8**, 565.
LONGUET-HIGGINS, M. S. & STEWART, R. W. 1961 *J. Fluid Mech.* **10**, 529.